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1980 J. Phys. A: Math. Gen. 13 3339

(http://iopscience.iop.org/0305-4470/13/10/028)

J. Phys. A: Math. Gen. 13 (1980) 3339–3343. Printed in Great Britain

Lower-bound RSRG approximation for a large n system

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Received 19 February, in final form 21 April 1980

Abstract. The lower-bound approximation for the real-space renormalisation group can be formulated for an n-component generalised Gaussian model in d dimensions. In this paper, this approximation is studied with the aid of a $1/n$ expansion, which is applied to the approximate recursion equations. Critical behaviour, including the index ν, thus calculated is compared with the exactly known $1/n$ expansion.

In recent years, real-space renormalisation group transformations have proven to be a valuable tool for developing a qualitative understanding of critical phenomena. The lower-bound method of Kadanoff (1975) and Kadanoff *et al* (1976; to be referred to as KHY) has been extensively employed despite the fact that we have no firm knowledge about its range of validity. This paper is designed to test the lower-bound approximation by applying it to a system with a large number of components, $n \to \infty$. Since, in this limit, the behaviour of critical systems is known (see, for example, the review paper by Ma (1976)), one can use this calculation to test the renormalisation approximation.

In particular, we focus upon the critical index ν. For large n one can expand ν in a series in $1/n$. The lowest order term agrees with Stanley's (1968) observation that as $n \to \infty$, the critical properties reduce to that of the spherical model (Berlin and Kac (1952)). When our calculation is carried to next order, however, we see some discrepancies between the exact expansion (see figure 1) and the $1/n$ term in the approximately calculated ν^{-1} which is zero for dimensionality two and four and reaches a maximum near $d = 3$.

Let us consider the generalised Gaussian model. The lower-bound approximation has been applied to this in the case of small $\epsilon = 4 - d$ and also in the case of the system with one spin component for $2 < d < 4$ (KHY). The form of the Hamiltonian is

$$H = \tfrac{1}{2} \sum_r \boldsymbol{\sigma}(r)^2 - \sum_R v(\boldsymbol{s}(R)^2) \tag{1}$$

where $\boldsymbol{\sigma}(r)$ is a spin vector with n components and defined at the points r of a simple hypercubic lattice in d dimensions. $\boldsymbol{s}(\boldsymbol{R})$ is the total spin on the hypercube designated by $\boldsymbol{R}$. In KHY, the approximate renormalisation group transformation for the potential $v(s^2)$ is obtained using the potential moving technique. This transformation involves a variational parameter p, which is proposed to be fixed by maximising the free energy. Instead, here the value of p is set to be $\frac{1}{2}$ which gives $\eta = 0$, while other values of p would give $\eta < 0$. The reason for the simplicity in the large-n limit lies in the fact that though each spin fluctuates a lot, the sum of the squares of the n spin components is a large number of $O(n)$ and relative fluctuation is small as $1/\sqrt{n}$. To take advantage of this situation, we write the potential as

$$v(s^2) = (s^2/2z^2) + u(s^2 - \tilde{n}) \tag{2}$$

where $z = 2^d$.

The first term in equation (2) is explicitly extracted because it balances the Gaussian term in the Hamiltonian (1). Our interest is in the reduced potential $u(s^2 - \tilde{n})$. The value of $\tilde{n}$ is determined such that $s^2 = \tilde{n}$ gives the minimum of the reduced potential, i.e.,

$$\mathrm{d}u(s^2 - \tilde{n})/\mathrm{d}s^2|_{s^2=\tilde{n}} = 0. \tag{3}$$

Although s^2 itself is an order of n, we can expect $s^2 - \tilde{n} \sim O(\sqrt{n})$ because $s^2 - \tilde{n}$ is a fluctuation around the minimum point of the reduced potential. Actually, as we will see later, $u(s^2 - \tilde{n}) \sim O(1)$.

In terms of the reduced potential defined in equation (2) the renormalisation group transformation of KHY is written in a relatively simple form:

$$u'(m^2 - \tilde{n}') = \ln\langle\exp(zu(s^2 - \tilde{n}))\rangle_{m^2} + \text{constant}. \tag{4}$$

Here, the lattice constant has been doubled and u, s^2 and $\tilde{n}$ have been transformed into u', m^2 and $\tilde{n}'$, respectively. The average $\langle\ \rangle_{m^2}$ is defined by

$$\langle f(s^2)\rangle_{m^2} = \frac{\int \mathrm{d}^n s\, f(s^2) \exp\{-[\boldsymbol{s} - (2/z^{1/2})\boldsymbol{m}]^2/4z\}}{\int \mathrm{d}^n s\, \exp\{-[\boldsymbol{s} - (2/z^{1/2})\boldsymbol{m}]^2/4z\}} \tag{5}$$

and depends on m^2. The new reduced potential u' is required to have a minimum at $m^2 = \tilde{n}'$:

$$\mathrm{d}u'(m^2 - \tilde{n})/\mathrm{d}m^2|_{m^2=\tilde{n}'} = 0. \tag{6}$$

In the $n \to \infty$ limit, s^2 in equation (4) can be replaced by its average $\langle s^2\rangle_{m^2} = 2nz + 4m^2/z$ and equation (4) is

$$u'(m^2 - \tilde{n}') = zu(2nz + 4m^2/z - \tilde{n}) + \text{constant}. \tag{7}$$

Then equations (3), (6) and (7) give the relation between $\tilde{n}'$ and $\tilde{n}$

$$\begin{aligned} \tilde{n}' &= \tfrac{1}{4}z(\tilde{n} - 2nz) \\ &= 2^{d-2}(\tilde{n} - 2^{d+1}n). \end{aligned} \tag{8}$$

This implies

$$1/\nu = d - 2 + O(1/n) \tag{9}$$

in agreement with exact calculations in the lowest order $1/n$ expansion. In this limit $n \to \infty$, the parameter $\tilde{n}$ decouples from other parameters and actually is the relevant scaling field. In the next order in $1/n$, however, a wider class of parameters must be taken into account.

Let us write the reduced potential as

$$zu(x) = -ax^2 - bx^3 + \text{higher terms} \tag{10}$$

where $x = s^2 - \tilde{n}$. There is no linear term because of equation (3) and higher terms do not contribute to the $O(1/n)$ correction to the critical exponent. The parameter space of $\{\tilde{n}, a, b\}$ is large enough to determine the correction and we will see that $\tilde{n} \sim$

$O(n)$, $a \sim O(1/n)$ and $b \sim O(1/n^2)$. Equation (4) is written to the desired order as

$$u'(m^2 - \tilde{n}') = \ln\langle(1 - bx^3)\exp(-ax^2)\rangle_{m^2} + \text{constant}. \tag{11}$$

The average in equation (11) cannot be calculated exactly because x^2 in the exponent is actually quartic in s. However, if we use an identity

$$\exp(-ax^2) = \int_{-\infty}^{\infty} d\lambda\ \exp(-\lambda^2/2a + i\sqrt{2}\lambda x)$$

the average can be written as

$$\begin{aligned}
&\langle\exp(-ax^2)\rangle_{m^2} \\
&\quad = \int_{-\infty}^{\infty} d\lambda\ \exp(-\lambda^2/2a)\langle\exp(i\sqrt{2}\lambda x)\rangle_{m^2} \\
&\quad = \int_{-\infty}^{\infty} d\lambda\ \exp(-\lambda^2/2a)\exp\Big[-i\sqrt{2}\tilde{n}\lambda + \frac{m^2}{z^2}\Big(\frac{1}{1 - i4\sqrt{2}z\lambda} - 1\Big) \\
&\quad\quad - \frac{n}{2}\ln(1 - i4\sqrt{2}z\lambda)\Big].
\end{aligned} \tag{12}$$

This integration can be calculated to any desired order by expanding the second exponent with respect to λ because we expect $\lambda^2 \sim a \sim O(1/n)$. To our desired order we have

$$\langle\exp(-ax^2)\rangle_{m^2} = (aA)^{-1/2}\exp(-\alpha^2/A) \tag{13}$$

where $A = 1/a + 64m^2 + 16nz^2$ and $\alpha = 4m^2/z + 2nz - \tilde{n}$. The average $\langle x^3\exp(-ax^2)\rangle_{m^2}$ is conveniently calculated by differentiating equation (13) with respect to $\tilde{n}$:

$$\begin{aligned}
&\langle x^3\exp(-ax^2)\rangle_{m^2} \\
&\quad = \frac{1}{8a^3}\Big[6a\frac{\partial}{\partial\tilde{n}}\langle\exp(-ax^2)\rangle_{m^2} + \frac{\partial^3}{\partial\tilde{n}^3}\langle\exp(-ax^2)\rangle_{m^2}\Big] \\
&\quad = (aA)^{-1/2}(1/8a^3)[12a\alpha/A - 12\alpha/A^2 + 8(\alpha/A)^3]\exp(-\alpha^2/A).
\end{aligned} \tag{14}$$

The transformation of the reduced potential is obtained by substituting equations (13) and (14) into equation (11):

$$\begin{aligned}
u'(m^2 - \tilde{n}') = &-\alpha^2/A - \tfrac{1}{2}\ln(aA) + \ln\{1 - (b/8a^3)[12a\alpha/A - 12\alpha/A^2 + 8(\alpha/A)^3]\} \\
&+ \text{constant}.
\end{aligned} \tag{15}$$

The transformed potential $u'(m^2 - \tilde{n}')$ can be written in an analogous form to equation (10) in terms of transformed parameters a' and b':

$$zu'(m^2 - \tilde{n}') = -a'(m^2 - \tilde{n}')^2 - b'(m^2 - \tilde{n}')^3 + \text{higher terms}. \tag{16}$$

The new parameters n', a' and b' are given by equation (6),

$$a' = -\frac{z}{2}\frac{\partial^2}{\partial(m^2)^2}u(m^2 - \tilde{n}')\Big|_{m^2=\tilde{n}'} \quad \text{and} \quad b' = -\frac{z}{6}\frac{\partial^3}{\partial(m^2)^3}u(m^2 - \tilde{n})\Big|_{m^2=\tilde{n}'}.$$

The renormalisation group transformation to $O(1/n)$ is

$$\frac{4\tilde{n}'}{z} - \tilde{n} + 2(n+2)z + \frac{3b}{4a^2}\left(1 - \frac{1}{\tilde{A}}\right) - 6(4z)^3\left(\frac{\tilde{n}'}{z^2} + \frac{n}{6}\right) = 0$$

$$a' = \frac{16}{z}\frac{a}{\tilde{A}} \tag{17}$$

$$b' = -64\frac{16}{z}\frac{a^2}{\tilde{A}^2} + 8\times 64^2\frac{a^3 z}{\tilde{A}^3}\left(\frac{\tilde{n}'}{z^2} + \frac{n}{6}\right) + \frac{1}{4}\left(\frac{16}{z}\right)^2\frac{b}{\tilde{A}^3}$$

where $\tilde{A} = 1 + 16\, a(4\tilde{n}' + nz^2)$.

The fixed points are

$$\tilde{n}^* = \frac{2nz}{z-4} + O(1)$$

$$a^* = (16/z - 1)(z-4)/16nz^2(z+4) + O(1/n^2) \tag{18}$$

$$b^* = -a^{*2}\frac{4z[1 - 3(1 - z/16)(z+8)/(z+4)]}{(1 - z/64)} + O(1/n^3).$$

The critical exponent ν to $O(1/n)$ is the eigenvalue of the linearised transformation around these fixed points.

$$\frac{1}{\nu} = d - 2 + \frac{4}{n\ln 2}\frac{1 - z/16}{64 - z}\frac{z-4}{z+4}\left[\!\left[3z + 4\frac{z^2 + 16z - 32}{z+4}\left(1 - \frac{z}{64}\right) - \left(1 - \frac{z}{16}\right)\left\{\frac{z(z+8)}{z+4} + \frac{16}{15z}\left[\frac{19}{2}(64 - z) + \frac{(64-z)(9z-64)}{16}\frac{z+8}{z+4} - \left(3 - \frac{(1 - z/16)(z+8)}{z+4}\right)[5(32 - z) - 4(z-4)]\right]\right\}\right]\!\right] + O(1/n^2). \tag{19}$$

The $1/n$ correction to this critical exponent is shown in figure 1 for $2 < d < 4$ and is compared with the exact result in $O(1/n)$.

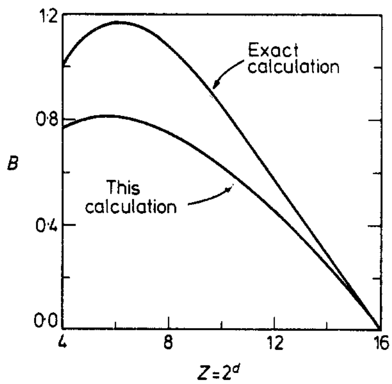

Figure 1. $1/n$ correction to ν plotted against the coordination number z of the hypercubic lattice. $\nu = 1/(d-2) - (1/n)B$. Since $\nu^{-1} = (d-2)[1 + (1/n)(d-2)B] + O(1/n^2)$ the error in ν^{-1} vanishes at both $d = 2$ and $d = 4$. The exact result is obtained from the scaling law $\nu = \gamma/(2-\eta)$ and exact calculations for η and γ to the first order in $1/n$.

It is already known that the lower-bound approximation is quite accurate for $n = 1$(KHY). In this paper, we have shown that it has a semi-quantitative accuracy for large n. It may well be useful, then, for a variety of studies of intermediate values of n.

Acknowledgments

We thank A Brown for checking our results by numerical calculations. We also thank T Eguchi for reading the manuscript.

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